THE CHARACTERISTICS OF INERTIAL RESONANCES IN A ROTATING LIQUID*

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We consider oscillating vortices induced in a rotating liquid by excitation of its free oscillations [1-3]. The conditions for vortex formation, both quantitative and qualitative, depend on the specifics of excitation and on which modes result for the given excitation source and geometry of the container. Nevertheless the hope is to find some general features of the evolution of the flow induced by the excitation of different free oscillations and perhaps to be able to predict the formation of vortices in a rotating liquid for different initial perturbations. The present paper describes the first experimental search for these general features.

The apparatus is shown schematically in Fig. 1. A water-filled and hermetically sealed transparent cylindrical container 1 is mounted vertically on a rotating platform coaxial with the container. The bottom of the container 2 is made of elastic resin. Free oscillations of the rotating liquid are excited by a generator consisting of hemispheres 3 mounted on a disk 4. The generator is supported on a shaft 5 coaxial with the container and which can rotate relative to the container and can move in the vertical direction.

The liquid in the container was first brought into rigid-body rotation with angular velocity Ω and the generator of the oscillations was brought into a steady state with angular velocity ω. Then the generator was raised upward by a height h from the position where the hemispheres touch the resin membrane bottom. The hemispheres therefore pressed into the bottom and produced bulges on it, which move with respect to the container with the required velocity. This method of exciting inertial oscillations was used in [4]. It has an advantage compared to other methods in that it avoids direct viscous interactions between the generator of the excitations and the liquid and therefore does not create undesirable secondary flows.

The flow field was studied by visual observation. The flow was made visible by adding to the water polymer spheres with small negative buoyancy (ρ = 1.00-1.05 g/cm^3) of radii less than 1 mm.

With the above method of exciting inertial oscillations, if we do not attempt to describe the exact form of the elastic bottom, the problem is characterized by eight parameters. They are the parameters Ω, ω, and h introduced above, the height H and radius R of the container, the kinematic viscosity ν of the liquid, the number of perturbing bodies M and their distance b from the axis of the system. Note that the elastic nature of the resin membrane and the radius of curvature of the perturbing bodies do not appear in the above set of parameters. In addition it is assumed that for M > 1 all bodies are located at the same distance from the axis. From the above minimum set of eight parameters a total of six dimensionless independent parameters can be constructed: H/R, b/R, h/R, M, the relative excitation frequency $f = 1 - \omega/\Omega$ and the Reynolds number Re, which will be discussed below.

The free oscillations of the rigidly rotating liquid satisfy linearized equations of motion and can be written as combinations of normal modes [5]:

$$v = I_m(\alpha_{mj}r/R)\sin(k_{mj}z/H)\exp(im(\theta-\omega t)),\ k_{mj} = \alpha_{mj}\frac{H}{R}\left[\frac{4}{m^2f^2}-1\right]^{-1/2}. \quad (1)$$

Here r, θ, z are the usual cylindrical coordinates, v is the z-component of the velocity, ω is the frequency of the wave, m = 1, 2,... is the angular wave number, I_m is the modified Bessel function of the first kind of order m, j is the number of zeros of the radial component of the velocity on $0 < r \leqslant R$, and $f = 1 - \omega/\Omega$. The condition of nonpenetration of fluid at r = R leads to an equation for $\alpha \equiv \alpha_{mj}$

*Devoted to the memory of Valery Fedorovich Tarasov, close colleague and teacher.

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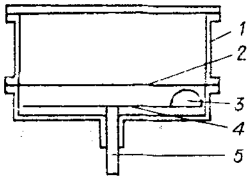

Fig. 1

$$2I_m(\alpha) + f\alpha \frac{dI_m(\alpha)}{d\alpha} = 0,$$

which together with (1) determine the dependence $\omega(k)$ ($k \equiv k_{mj}$). The condition of nonpenetration at $z = 0$ and $z = H$ implies that an integral number n of half-waves must fit within the height of the container, i.e., $k = \pi n$ ($n = 1, 2, \ldots$). The set of three integers (m, j, n) specifies the normal mode with dimensionless relative frequency $f_{mjn} = 1 - \omega_{mjn}/\Omega$. For all modes $|f_{mjn}| < 2/m$.

The following fixed parameters were used in the experiments: R = 25 cm, b/R = 0.78, and the radius of curvature of the hemispheres was 4.5 cm. The fundamental modes of excitation were (3, 1, n) with n = 1-5 and f > 0. The restriction on R was due to technical factors but the other restrictions were adapted only to reasonably limit the scope of the study. The study of the problem for other values of the parameters will be the subject of future experimental work.

Oscillating vortices analogous in structure and behavior to those described in [1-3] formed when the above modes were excited in the liquid. But differences were observed, obviously resulting from the geometry of the inertial wave. For example, in [1] vortices tended to form in the center of the container whereas in our case they formed near the sides of the container. This is not unexpected, since it was pointed out in [1] that vortices form in the regions of maximum vertical velocity in the wave. The maxima for the (3, 1, n) modes lie near the sides of the container. Another difference is the tilt of the axis of the vortex and its precession with the frequency of the wave. This effect is most extreme for small wavelength and results from the fact that different parts of the vortex are carried off differently by the wave. For excitation of modes with n > 1, n-member vortices are formed where neighboring half-wave sections of the vortex axis are tilted in opposite directions with respect to the vertical, recalling a "snake". The motion of the liquid in the vortices is completely analogous to that described in [1].

The primary interest of the work is the dependence of the formation time of an oscillating vortex on the different parameters of the problem and the conditions for which perturbations of the rotating liquid do not lead to strong vortices, since the results of the study might be usable for tornado prediction. It was shown in [1, 2] that tornadoes may form when free oscillations of a parent mesocyclone are excited by different factors, such as active convection, deformation of the mesocyclone by a background shear flow, atmospheric wave motion, and the interaction of the mesocyclone with irregularities on the earth's surface. In order to be able to predict tornadoes reliably it must be determined if tornadoes can form when a particular mesocyclone interacts with a particular topography and if so, how much time is required? The time required for the formation of a vortex is a very important parameter, since in natural conditions the time of interaction of a mesocyclone with mountains or hills on the earth's surface is limited because of the motion of the mesocyclone. After learning how to estimate the vortex formation time for an arbitrary mesocyclone and knowing the direction and speed of the latter, one may be able to more accurately predict the time and place of formation of a tornado. Hence the main focus in the present paper is to find the conditions for the formation of vortices and the vortex formation time.

A difficulty in measuring the time to form oscillating vortices is that there do not exist methods of measuring and analyzing the type of extremely inhomogeneous and unsteady three-dimensional flow produced in the experiments. The only available method was visual observation, which is simple and effective, but subjective.

The essence of the method is briefly as follows. By means of multiple observations of vortex formation the clearest and most reliably detectable phase was identified and the time required for this phase to develop from the time of introduction of the perturbation in the fluid was measured with a stopwatch. This phase was taken to be the instant when the vortex first appears as a local twisted jet with obvious retrogressive motion. Obviously this definition is not completely objective because it does not include any quantitative characteristics and it is probably difficult to understand by someone who has never observed the formation of an oscillating vortex. However, excitation of different inertial modes produces vortices different in size, structure, intensity, and oscillation frequency. A simple unified quantitative criterion cannot be established without at the same time losing the qualitative similarity.

To increase the reliability of the measurements of the formation time of oscillating vortices, a double timing method was used in which the instant at which the required phase of a nascent vortex appears is measured first and then the time at which the existence of at least one vortex is certain is measured next. A small difference in the two measured times indicates satisfactory results and the amount of work to obtain data for the statistical analysis of the data can be reduced. Another advantage of this technique is that pronounced beats occur in the flow when the difference between the excitation frequency and the required normal frequency is small. The beats show up as follows. After introducing the perturbation one first observes vortex precursors of different durations in which one or in more vortices form and complete several oscillations. Then a pause occurs during which they do not oscillate and seem to practically disappear. After the pause the vortices form again and reach high intensities. In a number of cases vortex precursors lasting approximately one vortex oscillation period were observed. In this case the second measured time often was quite later than the first and either the excitation frequency was corrected or the measurements were taken more carefully or a large number of measurements was taken for the statistical analysis of the data.

As noted above, the main focus of the present paper is the vortex formation time. As a function of the excitation frequency f this quantity has a sharp local minimum at the resonance point. It was first necessary to determine how accurately must the required frequency f be maintained and how stable must it be in order to obtain reliable results. Therefore, we first measured the dependence of the resonance region for the different modes on h/R. The resonance region of mode (m, j, n) is defined as the interval of excitation frequencies f about the natural frequency f_{mjn} inside which the vortex formation time varies by not more than one period of oscillation of the vortex. We note that random measurement errors are determined by the behavior of the vortex and by the details of the detection method and in most cases are of the same order, although there were cases of very reliable detection of vortices, when the measurement errors were much smaller.

Figure 2 shows the experimental dependence of the width of the resonance region Δf on h/R for different modes and experimental conditions: a) mode (3, 1, 1), H = 18.4 cm, T = 2.99 sec, M = 3; b) mode (3, 1, 1), H = 18.4 cm, T = 3.00 sec, M = 1; c) mode (3, 1, 1), H = 8.6 cm, T = 4.81 sec, M = 3; d) mode (3, 1, 3), H = 18.4 cm, T = 2.99 sec, M = 3; e) mode (3, 1, 3), H = 18.4 cm, T = 4.81 sec, M = 3; f) mode (3, 1, 2), H = 18.4 cm, T = 2.99 sec, M = 3. The hemispheres on the generator disk were separated by 120° which is the optimum arrangement for excitation of modes of the third angular harmonic.

We conclude from the data of Fig. 2 that for sufficiently large h the width of the resonance region Δf increases linearly with increasing h/R. The straight lines on graphs a-f were constructed from 6, 4, 3, 5, 3, and 4 points, respectively. At small h/R the dependence deviates from the linear law and in cases d-f the width of the resonance region actually increases with decreasing h/R. In simple oscillating systems this effect occurs because of energy dissipation and so it is reasonable to suppose that viscous friction is the cause here as well.

The effect of viscosity on the excitation of inertial waves is parametrized by the Reynolds number for the problem. We start from the formula $Re = u\ell/\nu$, where ν is the kinematic viscosity, u is the typical velocity, and ℓ is the minimum characteristic length scale over which the velocity u occurs. For the modes considered here we take $\ell = \lambda/2$ ($\lambda/2 = H/n$ is half the wavelength) and as u we take the velocity of the fluid around the mound produced on the bottom of the container, which is $f\Omega b$ in the azimuthal direction. Since the flow can be inviscid, the translational velocity induced by the perturbation is approximately $f\Omega bMh/R$. We finally obtain the Reynolds number $Re = f\Omega bMh\lambda/(2\nu R)$ which involves the characteristics of both the perturbation and the wave.

It follows from Fig. 2 that the behavior of the width of the resonance region becomes anomalous when the Reynolds number Re constructed above is less than the critical value $Re_* = (2.3 \pm 0.2)\cdot 10^5$. For Re less than approximately $0.3Re_*$ oscillating vortices do not occur in the flow because of viscous suppression of the inertial mode.

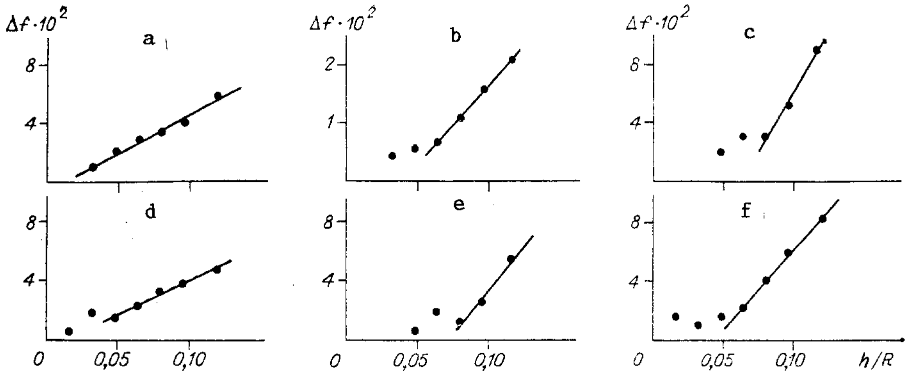

Fig. 2

Taking into account the excitation method and the universality of Re_*, one can construct for each inertial mode the quantity h_*/R bounding the region of linear dependence of Δf on h/R from the left. The quantity h_*/R is 0.025 ± 0.002; 0.073 ± 0.007; 0.059 ± 0.006; 0.05 ± 0.005; 0.079 ± 0.008; 0.036 ± 0.004 for cases a-f of Fig. 2, respectively. For a-e these values accurately reflect the transition to subcritical dependence, while for case f the calculated value of h_*/R appears to be a bit too small. In the opinion of the author this disagreement is not caused by errors in the above discussion, but is associated with the nature of the excitation of the (3, 1, 2) mode. Note that for cases a and d, with similar experimental conditions, the slopes of the straight lines are similar (0.52 and 0.45), whereas for case f the slope is more than twice as large (1.1) for the same conditions. The anomalous behavior of modes with an even number of half-wavelengths will be discussed further below.

The Reynolds number defined above and its critical value are universal for all modes with an odd number of half-wavelengths n. But the parameter space of the problem has not been fully explored and the formula for Re and hence the critical value will be corrected below.

The Re number introduced above is mode-dependent, since it can be constructed for each of the possible modes using the mode characteristics. Even assuming that the perturbation excites all modes equally, Re will be very different for different modes. Since Re determines the degree of viscous suppression of the mode, for a fixed amplitude and excitation frequency (h and f) only a finite set of modes is actually excited and this set can be very limited because modes with high m are not excited because of the restriction $|f_{mjn}| < 2/m$ and modes with high j and n are suppressed by viscosity, since the oscillation cell is small. Because the number of excited modes is therefore small, conditions can be found such that mode mixing practically does not occur; this was done here.

The dimensionless vortex formation time t/T also depends on Re. This dependence was studied experimentally. Only the velocity of rotation of the container was varied and ν was fixed (water) and h/R = 0.064. The results are shown by the points in Fig. 3 for the (3, 1, 1) mode with λ/2 = 8.6, 12.3, 18.4, 27.35 cm and natural frequency f = 0.578, 0.514, 0.412, 0.304 for a-d, respectively.

The data shown in a-d of Fig. 3 can be approximated by the straight lines $\log(t/T) = -0.387\log Re + 2.42$; $-0.312\log Re + 2.16$; $-0.273\log Re + 1.90$; $-0.344\log Re + 2.23$, which accurately reproduce the dependence of the vortex formation time on Re in the region studied. The coefficients were close for all of the lines but they will probably be different for different h/R, although the approximate dependence $t/T \sim Re^{-1/3}$ will probably hold in this region of Re. Unfortunately, it was not possible to confirm this or to find the dependence of the coefficients of the straight lines on h/R because of the narrowness of the Reynolds-number region assumed. The Re region is bounded from below by the critical value Re_* and is bounded from above by technical problems which would be difficult to overcome for the experimental setup used here.

The next step is to find the dependence of the vortex formation time on the number of (identical) bodies used to excite waves. Measurements were taken for the (3, 1, 1) mode in a container with H = 18.4 cm (H/R = 0.736, f = 0.412). The number of bodies M did not exceed the number of the angular harmonic m = 3. The bodies were arranged at the same radius from the axis of the container and separated by 120°. In some of the runs one or two

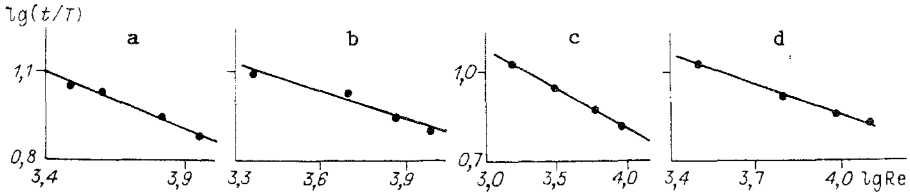

Fig. 3

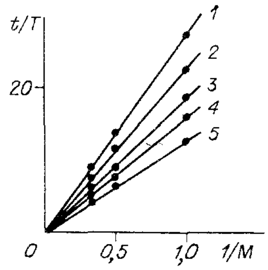

Fig. 4

of the bodies were removed from the generator. All of the runs were done for T = 2.99 sec. The vortex formation time was found to be proportional to 1/M for excitation of the n = 1 mode. This result is illustrated in Fig. 4. Since for each group of three points the straight line constructed using least squares passes through the origin to within experimental error, the lines shown on the graph are constructed using least squares but are constrained to pass through the origin. Lines 1-5 correspond to excitation amplitudes h/R = 0.048; 0.064; 0.08; 0.1; 0.12. We note that the vortex formation time should increase if the hemispheres are arranged on the disk with a different azimuthal angular separation and the other conditions are the same. However, the dependence on the position of the bodies was not considered in the present paper.

The quantity H/R is a dimensionless parameter of the problem. It would be interesting to know its effect on the vortex formation time. However, an arbitrary change in H/R leads to a different set of natural frequencies and for fixed f the resonance conditions for different H/R either do not occur simultaneously or occur for waves with completely different geometries, which would strongly affect the vortex formation time. Hence, measurements of this kind would be pointless because it would not be possible to give a clear interpretation to the results. However, these difficulties can be avoided if for a fixed f, m, j we choose the height H such that inertial mode (m, j, n) with frequency f_{mjn} = f occurs in the container. Then the modes will differ only by the number n of geometrically identical half-wave sections and the other factors will have no affect on the vortex formation time. Using this method three series of runs for (3, 1, n) modes were performed to determine the vortex formation time as a function of n for different perturbation amplitudes h: 1) f = 0.304, λ/2 = 27.3 cm, n = 1-3; 2) f = 0.514, λ/2 = 12.3 cm, n = 1-5; 3) f = 0.618, λ/2 = 6.1 cm, n = 1-3. In all runs the period of rotation of the container was T = 3.0 sec. For odd n the following result was obtained. For a given h/R the dimensionless vortex formation time t/T was less for n = i than for n = i + 2 by a quantity 2Δ independent of h/R and hence Re. The value of Δ per half-wavelength was calculated for each of the three cases as the average $\langle t_{i+2} - t_i \rangle /2$, where t_i is the dimensionless vortex formation time (t/T) for n = i. The difference was calculated using the t_i values obtained experimentally for the same values of f, λ, and h/R, but different n. Then the average was calculated over groups of differences obtained for the same f and λ, successive odd values of n, and different h/R. The resulting Δ values for cases 1-3 are 3.48 ± 0.17; 0.88 ± 0.13; and 0.89 ± 0.06, respectively.

The difference in the vortex formation times for modes with n differing by unity is in general not the same for different h/R. However, in case 3 it is approximately the same, although the error is somewhat larger (Δ = 0.89 ± 0.23). This tendency was not observed for longer wavelength modes, which once again illustrates the noted above fundamental difference between odd and even modes.

Finally, we summarize the basic results of the present paper.

1. We have constructed a Reynolds number for waves determining the viscous selection of inertial modes because of the suppression of small-scale oscillations. It leads to a sufficient condition for the nonappearance of oscillating vortices in a perturbed rotating

liquid, since if the largest mode number is less than the threshold for vortex formation then none of the free oscillation modes can produce vortices and therefore they will not exist.

2. The dependence of the vortex formation time on the Reynolds number, number of perturbing bodies, and the number n of half-wavelengths has been determined. The data show that if we extrapolate the vortex formation time into the region of large n and Re, the expected vortex formation time will be less than the time necessary for arrival of the reflected wave. This implies that a vortex can form not only in a standing wave, but also in a traveling wave and the existence of a reflecting surface is not essential. This tentative prediction requires direct experimental confirmation, but it implies the following for the atmosphere:

1) when a mesocyclone interacts with an obstacle, a tornado can form after a time smaller than the period of rotation of the mesocyclone; for typical mesocyclone translational velocities an interaction time of this order can be guaranteed, even in the case of a small hill (change in height over the radius of the mesocyclone of 100-500 m). A vortex is formed within a few kilometers of it, although the vortex is not necessarily destructive, since the factors determining the intensity of a vortex have not yet been studied;

2) although multi-member vortices ($n > 1$) are possible in nature, they are less probable because they require much more time to form;

3) the presence of a reflecting inversion layer (the analog of the cover in the experiment) in the troposphere is not essential for vortex formation.

The experimental data presented here obviously are not exhaustive for the problem. Further progress in the solution of this problem is outside the scope of this single paper.

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